Domination of Vertices in a Graph

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Abstract

In a graph G = (V, E), a dominating set is a subset S of V such that every vertex in $V(G)\setminus S$ is adjacent to at least one vertex in S. The minimum cardinality among the dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$. We study the relation of domination number and some parameters such as independence, cover and matching.The domination number of a connected graph without isolated vertices is also bounded above by all of the covering and independence number.

Keywords: dominating set, domination number, path, diameter

Introduction

A graph G is a finite nonempty set V of vertices together with a possibility empty set E of 2-element subsets of V called edges. A graph G has vertex set V and edge set E, we write G = (V,E). If uv is an edge of G, then u and v are adjacent vertices (neighbors). A vertex is called isolated if it is not an endvertex of any edge. The set of neighbors of a vertex v is called the open neighborhood of vand is denoted by $N_G(v)$or N(v). The set N[v]=N(v)$\cup$ {v} is called the closed neighborhood of v.The number of vertices in a graph G is the order of G and the number of edges is the size of G. A walk of length k is an alternating sequence W= u_0, e_1, u_1, e_2, u_2, e_3, . . ., u_{k-1}, e_k, u_k of vertices and edges with $e_i = u_{i-1}u_i$. If all k +1 vertices are distinct, then W is called a path of length k. Agraph G is connected if for every pair u, v of vertices there exists at least one $u-v$ path, otherwise, G is disconnected. Thedistance d(u, v) between u and v is the minimum length of a $u-v$ walk. The eccentricity of vertex v is ecc(v) =max{d (v, w):w$\in$V}. The diameter of G diam(G) is max{ecc(v):v$\in$ V}. A graph H is a subgraph of a graph G if V(H)$\subseteq$(G)and E(H)$\subseteq$E(G), in which case we write H$\subseteq$G. For a nonempty subset S of V(G), the subgraph $\langle S\rangle$ of G induced by S has S as its vertex set and two vertices u and v are adjacent in $\langle S\rangle$ if and only if u and v are adjacent in G. A set S of vertices in a graph G is independent if no two vertices in S are adjacent. The maximum number of vertices in an independent set of vertices of G is called the vertex independence number of G and is denoted by $\alpha(G)$. The independent set S is maximal if S is not a proper subset of any independent set of G (Bondy, 1976).

Some Fundamental Concepts in Graph Theory Concerning a Set of Edges

The edges in an independent set of edges of G are called a matching in G. A matching of maximum size in G is a maximum matching in G. The edge independence number $\alpha'(G)$ of G is the number of edges in a maximum matching of G. Let M be a matching in a graph G. An M-alternating path of G is a path whose edges are alternating in M and not in M. An M-augmenting path is an M-alternating path P both of whose end-vertices are unmatched, that is, first and the last edges of P do not belong to M. A vertex and an edge are said to cover each other in a graph G if they are incident in G. The minimum cardinality of a vertex cover in a graph G is called the vertex covering number of a graph G and is denoted by $\beta(G)$. The edge covering number $\beta'(G)$ of a graph G without isolated vertices is the minimum cardinality

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of an edge cover in G.A set S of vertices of G is a dominating set if every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The minimum cardinality among the dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is then referred to as a minimum dominating set.A minimal dominating set in a graph G is a dominating set that contains no dominating set as a proper subset. A minimal dominating set of minimum cardinality is, of course, a minimum dominating set and consists of $\gamma(G)$ vertices (Chartrand & Lesniak, 2007) .

Some Properties on Domination of Vertices

Some properties of dominating set are

- Every minimum dominating set is a minimal dominating set.
- A graph can have many dominating sets.
- A dominating set may or may not be independent set.
- Loops or multiple edges do not affect domination.

Theorem: A dominating set S of a graph G is a minimal dominating set of G if and only if every vertex v in S satisfies at least one of the following two properties:(i) there exists a vertex w in $V(G) - S$ such that $N(w) \cap S = \{v\}$(ii)v is adjacent to no vertex of S.

Proof: Let S be a dominating set of G. For each vertex v in S has at least one of the properties (i) and (ii) as mentioned above, then $S - \{v\}$ is not a dominating set of G. Consequently, S is a minimal dominating set of G.

Conversely, assume that S is a minimal dominating set of G. Then for each $v \in S$, the set $S - \{v\}$ is not a dominating set of G. Hence there is a vertex w in $V(G) - (S - \{v\})$ that is adjacent to no vertex of $S - \{v\}$. If $w = v$, then v is adjacent to no vertex of S. If $w \neq v$. Since S is a dominating set of G and $w \notin S$, the vertex w is adjacent to at least one vertex of S. However, w is adjacent to no vertex of $S - \{v\}$. Consequently, $N(w) \cap S = \{v\}$ (Chartrand & Lesniak, 2007).

Theorem: If S is a minimal dominating set of a graph G without isolated vertices, then $V(G) - S$ is a dominating set of G.

Proof: For any vertex $v \in S$, then v has at least one of the two properties described in above Theorem. Suppose first that there exists a vertex w in $V(G) - S$ such that $N(w) \cap S = \{v\}$. Hence v is adjacent to some vertex in $V(G) - S$. Suppose next that v is adjacent to no vertex in S. Then v is an isolated vertex of the subgraph $\langle S \rangle$. Since v is not isolated in G, the vertex v isadjacent to some vertex of $V(G) - S$.Thus $V(G) - S$is a dominating set of G (Chartrand & Lesniak, 2007).

Algorithm on Domination Number of a Graph

Initially, set $D = \phi$,

Step 1: Include all the isolated vertices in D and remove all such vertices from the graph.

Step 2: Let a path, corresponding to the diameter of the resulting graph, exist between u_1 and u_n.

Step 3: Let $N(u_1) = \{u_{11}, u_{12}, \ldots, u_{1k}\}$ Choose u_{1p} such that $|N(u_{1p})| = \max\{| N(u_{1i}) | : i = 1, 2, \ldots, k\}$ and $diam(u_{1p}, u_n) \nless diam(u_{1i}, u_n)$ where $i = 1, 2, \ldots, k$ and $i \neq p$. Compare the cardinalities of $N(u_1)$ with $N(u_{1p})$. If $|N(u_1)| < |N(u_{1p})|$ then include u_{1p} in D, else include u_1 in D. Let the included vertex be α_1.

Step 4: If diam(G)> 3, then let $N(u_{nt}) = \{ u_{n1}, u_{n2}, \ldots, u_{nm}\}$. Choose u_{nt} such that $|N(u_{nt})| = \max\{|N(u_{ni})|, i = 1, 2, \ldots, m\}$ and $diam(u_1, u_{nt}) \nless diam(u_1, u_{ni})$ where $i = 1, 2, \ldots, m$ and $i \neq t$. Compare the cardinalities of $N(u_n)$ with $N(u_{nt})$. If $|N(u_n)| < |N(u_{nt})|$ then include u_{nt} in D, provided $u_{nt} \neq \alpha_1$, else include u_n in D. Let the included vertex be α_2.

Step 5: Let $B = N(\alpha_1) \cup N(\alpha_2)$. Delete all the edges having one end at either α_1 or α_2. Also delete all the edges having both ends in B. In this process, if any vertex becomes isolated, delete it from the graph.

Step 6: If the degree of any vertex in B is 1, then delete it from the graph.

Step 7: Repeat the steps 2-7, until the graph becomes an empty graph (Nagoorgian & Chandrasekaran, 2007).

Example: Consider the graph given in Figure 1(i). We will calculate the domination number ~~o~~f the graph for Figure 1(i) by using the above algorithm.

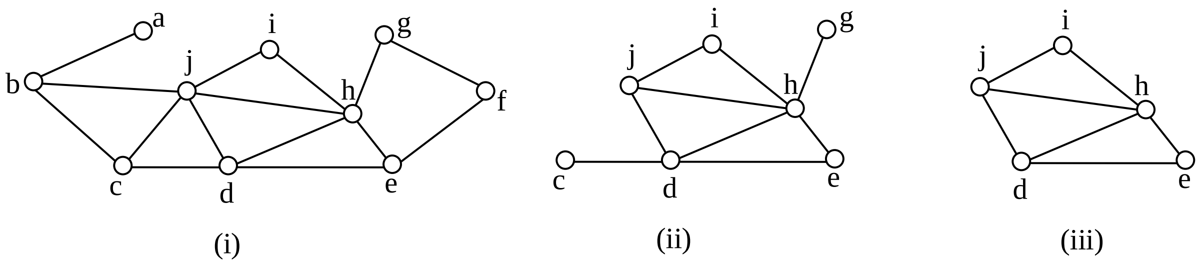

Figure. (1).

Step 1: There is no isolated vertices in Figure 1(i) and so $D = \phi$.

Step 2: The diameter of the given graph is 5 and a corresponding path is between vertex a and vertex f.

Step 3: Inspecting the neighbors of vertex a, it is easy to find that $\alpha_1 = b$ and $|N(a)| < |N(b)|$, so let D = {b}.

Step 4: Inspecting the neighbors of vertex f; it is easy to find that $\alpha_2 = f$. Include f in D and we have D = {b, f}.

Step 5: Now $B = N(b) \cap N(f) = \{a, c, j\} \cap \{e, g\} = \{a, c, e, g, j\}$. Delete all the edges having one end at either b or f. Also delete all the edges having both ends in B. The resulting graph is as shown in Figure 1(ii).

Step 6: In B, vertices c and g have only degree 1. Hence it is removed from the graph to get the graph in Figure 1(iii).

Step 7: For the resulting graph, we repeat Step 2 and a corresponding path is between vertex i and vertex d. Inspecting the neighbors of i, it is easy to find that $\alpha_3 = h$ and $|N(i)|<|N(h)|$. Therefore D = {b, f, h}.

Remove the edges incident the vertices b, f and h. We have B={a, c, d, e, g, i, j}. Also delete the edges having both ends in B. It is easy to findthat D becomes {b, f, h} and the graph becomes empty. Hence, for the given graph $\gamma(G) = 3$.

For graphs G without isolated vertices, an upper bound for $\gamma(G)$ in terms of the order of G is expressed in the following Corollary.

Corollary: If G is a graph of order n without isolated vertices, then $\gamma(G) \le \frac{n}{2}$.

Proof: Let S be a minimal dominating set of G. Then $V(G) - S$ is a dominating set of G. Hence $\gamma(G) \le \min\{|S|, |V(G) - S|\} \le \frac{n}{2}$ (Chartrand & Lesniak, 2007).

Theorem: Every graph G without isolated vertices contains a minimum dominating set S such that for every vertex v of S, there exists a vertex *w* of $V(G) - S$ such that $N(w) \cap S = \{v\}$.

Proof: Let S be a minimum dominating set of G such that $\langle S \rangle$ has maximum size. Suppose, to the contrary that S contains a vertex v that does not have the desired property. Then v is an isolated vertex in $\langle S \rangle$. Moreover, every vertex of $V(G) - S$ that is adjacent to v is adjacent to some other vertex of S. Since G contains no isolated vertices, v is adjacent to a vertex w in $V(G) - S$. Consequently, $(S - \{v\}) \cup \{w\}$ is a minimum dominating set of G and it's induced subgraph contains at least one edge incident with *w* and hence has a greater size than $\langle S \rangle$ which is a contradiction (Chartrand & Lesniak, 2007).

Theorem: If G is a graph without isolated vertices, then

$$\gamma(G) \le \min\{ \alpha(G), \alpha'(G), \beta(G), \beta'(G)\}.$$

Proof: Since every vertex cover of a graph without isolated vertices is a dominating set, as is every maximal independent set of vertices, $\gamma(G) \le \alpha(G)$ and $\gamma(G) \le \beta(G)$. Let X be an edge cover of cardinality $\beta'(G)$. Then every vertex of G is incident with at least one edge in X.

Let S be a set of vertices obtained by selecting an incident vertex with each edge in X. Then S is a dominating set of vertices and $\gamma(G) \le |S| \le |X| = \beta'(G)$.

Let M be a maximum matching in G. We construct a set S of vertices consisting of one vertex incident with an edge of M for each edge of M. Let $uv \in M$. The vertices u and v cannot be adjacent to distinct $\overline{M}$-vertices *x* and y, respectively; for otherwise,(x, u, v, y) is an M-augmenting path in G, contradicting to the fact that a matching M in a graph G is a maximum matching if and only if G contains no M-augmenting path. If u is adjacent to $\overline{M}$-

vertex, place u in S; otherwise, place v in S. This is done for edge of M. Thus, S is a dominating set of G, and $\gamma(G) \le |S| = |M| = \alpha'(G)$ (Chartrand & Lesniak, 2007).

Theorem: If G is a graph of order n and size m for which $\gamma = \gamma(G) \ge 2$, then(1) $m \le \frac{(n-\gamma)(n-\gamma+2)}{2}$.

Proof: The proof is by induction on n. When $\gamma(G) = 2$, the right side of (1) is $(n-2)n/2$. Since, we have $\Delta(G) \le n-2$. Hence $m = \frac{1}{2}\sum_{v\in V} \deg(v) \le \frac{1}{2}n(n-2)$ and the result follows. Now, suppose $\gamma(G) \ge 3$. When n = 3, $G = \overline{K}_3$. Hence m = 0and the inequality is trivially satisfied. We assume that the result is true for all graphs of order less than n. Let G be a graph of order n with $\gamma(G) \ge 3$ .Let $v \in V(G)$ and $\deg(v) = \Delta(G)$.

Then $|N(v)| = \Delta(G) \le n - \gamma(G)$. Let $|N(v)| = \Delta(G) = n - \gamma(G) - r$, where $0 \le r \le n - \gamma(G)$. Let $S = V - N[v]$.

Then $|S| = n - \Delta(G) - 1 = n - (n - \gamma(G) - r) - 1 = \gamma(G) + r - 1$.

Let m_1, m_2 and m_3 denote respectively the number of edges between N(v) and S, $\langle S \rangle$, and $\langle N[v] \rangle$. Now, for any $u \in N(v)$, the set $S_1 = (S - N(u)) \cup \{u, v\}$ is a dominating set of G. Hence $\gamma(G) \le |S - N(u)| + 2 \le \gamma(G) + r - 1 - |S \cap N(u)| + 2$.

Thus $|N(u) \cap S| \le r+1$ and hence $m_1 = \Delta(G)(r+1)$.

Now if D is a minimum dominating set of $\langle S \rangle$, then $D \cup \{u\}$ is a dominating set of G. Hence $\gamma(G) \le \gamma(\langle S \rangle) + 1$, so that $\gamma(\langle S \rangle) \ge \gamma(G) - 1 \ge 2$. By induction hypothesis, we have

$$m_2 \le \left[\frac{1}{2}\left(|S| - \gamma(\langle S \rangle)\right)\left(|S| - \gamma(\langle S \rangle) + 2\right)\right]$$

$$\le \left[\frac{1}{2}\left(\gamma(G) + r - 1 - (\gamma(G) - 1)\right)\left(\gamma(G) + r - 1 - (\gamma(G) - 1) + 2\right)\right] = \frac{1}{2}r(r+2).$$

Now the vertex v is adjacent to $\Delta(G)$ vertices in N(v) and for each vertex $u \in N(v)$, the number of edges between S and N(v) incident with u is at most $r+1$.

Hence $m_3 = |E(\langle N(v) \rangle)| \le \Delta(G) + \frac{1}{2}\Delta(G)(\Delta(G) - r - 2)$.(Chartrand & Lesniak, 2007)

Thus $m = m_1 + m_2 + m_3$

$$\le \Delta(G)(r+1) + \frac{1}{2}r(r+2) + \Delta(G) + \frac{1}{2}\Delta(G)(\Delta(G) - r - 2)$$

$$= \Delta(G)(n - \gamma(G) - \Delta(G) + 1) + \frac{1}{2}(n - \gamma(G) - \Delta(G))(n - \gamma(G) - \Delta(G) + 2)$$

$$+ \Delta(G) + \frac{1}{2}\Delta(G)(2\Delta(G) - n + \gamma(G) - 2),$$ where $r = n - \gamma(G) - \Delta(G)$

$$= \frac{1}{2}(n-\gamma(G))(n-\gamma(G)+2) - \frac{1}{2}\Delta(G)(n-\gamma(G)-\Delta(G))$$

$\le \frac{1}{2}(n-\gamma(G))(n-\gamma(G)+2)$.Hence the proof is complete by induction

(Chartrand & Lesniak, 2007).

Theorem: If G is a graph of order n and size m, then $n - m \le \gamma(G) \le n + 1 - \sqrt{1+2m}$.

Proof: By using the inequality (1) gives in above Theorem,

$$(n-\gamma(G))^2 + 2(n-\gamma(G)) - 2m \ge 0$$

$$n-\gamma(G) = \frac{-2 \pm \sqrt{2^2 - 4(1)(-2m)}}{2(1)} \ge 0$$

$$n-\gamma(G) = \frac{-2 \pm \sqrt{4+8m}}{2} \ge 0$$

$$n-\gamma(G) = \frac{-2 \pm 2\sqrt{1+2m}}{2} \ge 0$$

$$n-\gamma(G) = -1 \pm \sqrt{1+2m} \ge 0.$$

Solving the inequality for $n-\gamma(G)$ and using the fact that $n-\gamma(G) \ge 0$, we have that

$$n-\gamma(G) \ge -1 \pm \sqrt{1+2m} .$$

Hence $n-\gamma(G) \ge -1 + \sqrt{1+2m}$. Therefore $\gamma(G) \le n + 1 - \sqrt{1+2m}$. Since $\gamma(G) \ge 1$ the lower bound is established when $m \ge n-1$, which includes all connected graphs. Assume that $m \le n-1$. Then G is a graph with at least $n-m$ components. The domination number of each component of G is at least 1. So $\gamma(G) \ge n-m$. Thus $n - m \le \gamma(G) \le n + 1 - \sqrt{1+2m}$ (Chartrand & Lesniak, 2007).

Theorem: For a connected graph G, we have $\gamma(G) \ge \left\lceil \frac{\text{diam}(G)+1}{3} \right\rceil$.

Proof: Let u_0 and u_k be vertices of G of distance $d(u_0, u_k) = \text{diam}(G)$ from each other, and $p = (u_0, u_1, \ldots, u_k)$ be a path of length $k = \text{diam}(G)$. Without loss of generality, assume that G is simple.If D is a dominating set of G, then D must, in particular, dominate each vertex u_i on the path p. Clearly, every vertex u_i in D that lies on p can dominate at most three vertices on p: itself and at most its two neighbors on p, namely u_{i-1} and u_{i+1}. Also, every vertex in D that is not on p can dominate at most three vertices on p, which in addition must be consecutive on the path p, otherwise we have a shorter path between u_0 and u_k, contradicting the definition ofdiam(G).Therefore each vertex in D can dominate at most three vertices on p. Since the number of vertices of p is $k+1$, we have $3|D| \ge k+1$. Hence this inequality holds for all dominating set in D, and $|D|$ is an integer, the theorem follows (Geir & Raymond, 2007).

Conclusions

We study the domination and related parameter such as independence, cover and matching. The domination number of a connected graph without isolated vertices is also bounded above by all of the covering and independence number. The domination algorithm works fast for finding domination number. Dominating sets appear in many applications such as facility location problems, designs and analysis of communication networks, social network theory and routing problems.

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